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Cohomological reduction by split pairs

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Abstract

A new method is developed to compare cohomology in module categories of different rings. This method does in general not produce isomorphisms, but surjective (or injective) maps between extension groups of modules over the two rings involved. Applications of this method are given to abstract problems—we recover and extend results on the strong no loops conjecture—and to algebras naturally coming up in invariant theory—we relate the cohomology of Brauer algebras with that of various symmetric groups.

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0. Introduction

Suppose we are given two rings R and S and a ring homomorphism $f : R \to S$, and we would like to compare the cohomology in the category of, say, left R-modules with that of left S-modules. In general, nothing definitive can be said. There are, however, some situations, which have been studied intensively and successfully.

Assume that f is an embedding of rings (sending the unit of R to that of S). Still, nothing can be said—either ring could be semisimple without the other one being so. Assume also conditions like S being a projective R-module (via f). This is perfectly reasonable, for example, in the representation theory of finite groups, where we could have $R = kH$ and $S = kG$ for a subgroup H of a finite group G (and a field k). Then the machinery of induction and restriction functors will allow us to compare this cohomology, for example by a Mackey formula.

Another customary assumption is that f is surjective. Again, this is not enough—either ring could have finite global dimension without the other one having the same. But we can consider some additional conditions like the kernel of f being a projective R-module (at least on one side). This makes sense, for example in the representation theory of algebraic groups or of Lie algebras, when defining quasi-hereditary or stratified algebras. Then projective

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resolutions behave well under inflating S-modules to R-modules, and in good cases one gets a full embedding of derived categories $D^b(S\text{-mod}) \hookrightarrow D^b(R\text{-mod})$.

The aim of this article is to develop, and to apply, a new method of *comparing cohomology*, combining a subring situation with a quotient ring situation, but without assuming any of the strong conditions normally used in either of these situations. In particular, this method can be used to show the *non-vanishing of cohomology* in certain situations.

One feature of this approach is that it usually does not lead to isomorphisms in cohomology, but to surjective (or injective) maps between extension groups over the two rings involved. Thus, on the level of derived categories, we do not get embeddings in the usual sense (that is, injective on objects and bijective on morphisms). Instead, we get two exact (triangle preserving) functors F and G, going in opposite directions, such that G is injective on objects and injective on morphisms, and F, when restricted to the image of G, is surjective on objects and surjective on morphisms.

Having developed the general machinery, we then collect some evidence for this method to be practical and useful, both when dealing with abstract problems—we recover and extend a number of results in the literature, in particular on the *strong no loops conjecture*—and when studying algebras occuring in nature—we relate the cohomology of *Brauer algebras* with that of various *symmetric groups*.

This article is organized as follows:

In Section 1, we define the basic notions, split pairs and exact split pairs, and derive their basic properties.

The short Section 2 lists three elementary classes of examples. Section 3 then shows that all exact split pairs are, in some sense, made up of the three kinds of examples. Examples show that this classification cannot be strengthened.

In the remaining Sections 4 and 5, we demonstrate how to use the concept of (exact) split pairs, and at the same time we collect classes of examples of algebras to which our technology can be applied. In Section 4, after collecting various examples of Artinian or Noetherian algebras, we discuss in more detail another class of examples, Brauer algebras, which are of interest in the representation theory of reductive algebraic groups of types B and C. Such a Brauer algebra is shown to be related via split pairs to group algebras of various symmetric groups.

Section 5 looks at finite dimensional algebras given by quivers and relations. Various technical results produce split pairs relating an algebra $A = kQ/I$, with other algebras obtained from A by cutting vertices or arrows or larger parts of the quiver (always under certain assumptions, of course). As an application, we find new classes of algebras satisfying the strong no loops conjecture, which states that $\mathrm{Ext}^1_A(S, S) \neq 0$ for some simple A-module S implies that S has infinite projective dimension. Throughout this article, we restrict our attention to exact split pairs defined on the level of Abelian categories; these imply split pair situations on derived level, which may deserve to be studied separately.

We refer the reader to [2,5] for background material on rings and categories of modules, to [17] for homological algebra and to [11,12,15] for introductions to derived categories.

1. Definitions and basic properties

We begin this section by defining the basic structure we are going to use throughout the paper, the structure of an (exact) split pair of functors between two categories.

Exact split pairs will be used to compare the cohomology of two categories of modules. Indeed, an exact split pair of functors between two Abelian categories induces a split pair of (obviously triangulated) functors between the derived categories and hence relates the cohomology of the two categories; there are induced surjections and injections between Ext groups in the two Abelian categories. This allows us to compare Ext groups and numerical invariants associated, such as projective dimensions.

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. A pair (F, G) of additive functors $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ is a *split pair of functors* (between $\mathcal{A}$ and $\mathcal{B}$) if the composition $F \circ G$ is an autoequivalence of the category $\mathcal{B}$. If the categories are equipped with exact structures, and if the two functors are exact with respect to these exact structures, the split pair is called an *exact split pair of functors* (between $\mathcal{A}$ and $\mathcal{B}$).

Note that in this definition, the pairs $(\mathcal{A}, \mathcal{B})$ and (F, G) are ordered.

The definition of exact split pair of functors between two categories of modules can be reformulated, as for Morita equivalences, in terms of the existence of two bimodules.

In the proof, we are going to apply Watts' theorem [18]. Therefore, throughout this paper we need to *assume that rings are left Noetherian and modules are finitely generated. Moreover, functors send finitely generated modules to finitely generated modules.* Our main examples will be finite-dimensional algebras; in this case modules will be finite dimensional, too.

Alternatively we could assume one of the following: (a) Rings are arbitrary and modules are finitely presented; (b) Rings and modules are arbitrary, functors commute with arbitrary direct sums.

Lemma 1.2. *Let A and B be two left Noetherian rings. Denote the categories A-mod and B-mod (of finitely generated left modules) by $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the existence of an exact split pair of functors (F, G) between $\mathcal{A}$ and $\mathcal{B}$ is equivalent to the existence of two bimodules ${}_BT_A$ and ${}_AS_B$, each projective on the right, such that $T \underset{A}{\otimes} S$ is an invertible B-bimodule.*

Proof. If there are two such bimodules, then $F = {}_BT \underset{A}{\otimes} -$ and $G = {}_AS \underset{B}{\otimes} -$ are exact by right projectivity of S and T, respectively, and they form a split pair of functors by the assumption on $T \otimes S$.

Conversely, if (F, G) is an exact split pair of functors between $\mathcal{A}$ and $\mathcal{B}$, then by Watts' theorem (see [18], Theorem 2 and its proof, or Theorem 1 in case of the alternative assumption (b)), the right exact functors F, G are taking tensor products by bimodules ${}_BT_A$ and ${}_AS_B$, respectively.

That is, $F = {}_BT \underset{A}{\otimes} -$ and $G = {}_AS \underset{B}{\otimes} -$. The functors also being left exact, the bimodules T and S must be projective as right A and B modules, respectively.

Since the composition $F \circ G$ is a Morita equivalence from $\mathcal{B}$ to itself, the bimodule $T \otimes S$ must be isomorphic to B on the left and on the right. ■

Note that $F \circ G$ being an equivalence implies that $T \otimes_A S$ also is a projective generator on the right. The example of Morita equivalences shows that in general, $T \otimes_A S$ need not be projective as a bimodule.

Proposition 1.3. *If (F, G) is a split pair of functors between two additive categories $\mathcal{A}$ and $\mathcal{B}$, then F is surjective on the isomorphism classes of objects, and G is injective on the isomorphism classes of objects.*

Moreover, given objects $M, N \in \mathcal{B}$, the functor F induces an epimorphism of Abelian groups $\mathrm{Hom}_{\mathcal{B}}(G(M), G(N)) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(FG(M), FG(N)) \cong \mathrm{Hom}_{\mathcal{A}}(M, N)$ *on the morphism groups, and the functor G induces a monomorphism of Abelian groups* $\mathrm{Hom}_{\mathcal{B}}(M, N) \rightarrowtail \mathrm{Hom}_{\mathcal{A}}(G(M), G(N))$ *on the morphism groups.*

Proof. Since $F \circ G$ is an equivalence, any object M of $\mathcal{B}$ is isomorphic to an object of the form $(F \circ G)(N)$; that is, it is of the form $F(K)$ for $K = G(N)$. Moreover if M, N are two non-isomorphic objects in $\mathcal{B}$, then $(F \circ G)(M)$ and $(F \circ G)(N)$ are non-isomorphic (cf. [2, 21.1] for the Abelian case; their proof works for additive categories as well); thus $G(N)$ and $G(M)$ can't be isomorphic.

Now let M, N be objects in $\mathcal{B}$. Consider the group homomorphisms induced by F and G:

$$\mathrm{Hom}_{\mathcal{B}}(M, N) \xrightarrow{G^*} \mathrm{Hom}_{\mathcal{A}}(GM, GN) \xrightarrow{F^*} \mathrm{Hom}_{\mathcal{B}}(FGM, FGN).$$

Since $F^* \circ G^*$ is a group isomorphism ([5, Proposition 1.15]), the morphism F^* induced on the Hom groups by the functor F is surjective and the morphism G^* induced on the Hom groups by the functor G is injective. ■

Proposition 1.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Abelian categories. An exact split pair of functors (F, G) between $\mathcal{A}$ and $\mathcal{B}$ induces a split pair of triangulated functors (F_*, G_*) between the derived categories $D^b(\mathcal{A})$ and $D^b(\mathcal{B})$.*

Proof. The exact functors F, G induce triangulated functors $F_*\colon D^b(A\text{-mod}) \to D^b(B\text{-mod})$ and $G_*\colon D^b(B\text{-mod}) \to D^b(A\text{-mod})$ between the derived categories.

The inverse of $F_* \circ G_*$ is the functor Φ_* induced by the inverse Φ of $F \circ G$. ■

Corollary 1.5. *An exact split pair of functors (F, G) between two Abelian categories $\mathcal{A}$ and $\mathcal{B}$ induces, for $n \geq 0$, for M, N objects in $\mathcal{B}$, surjections* $\mathrm{Ext}^n{}_{\mathcal{A}}(GM, GN) \twoheadrightarrow \mathrm{Ext}^n{}_{\mathcal{B}}(M, N)$ *and injections* $\mathrm{Ext}^n{}_{\mathcal{B}}(M, N) \rightarrowtail \mathrm{Ext}^n{}_{\mathcal{A}}(GM, GN)$*. For $M = N$, these maps are ring homomorphisms of Yoneda algebras.*

In the terminology to be introduced in the next section, the last statement means that an exact split pair induces split quotients of Yoneda algebras.

Proof. Clear. ■

Therefore, homological properties in one Abelian category can be compared to such properties in another, possibly smaller, Abelian category, which may have 'less' cohomology. For example, we have the following inequalities between homological dimensions:

Corollary 1.6. *With the previous notation, one has* $pd(M) \leq pd(G(M))$, $id(M) \leq id(G(M))$, *for any object* M *in* $\mathcal{B}$, *and* $\mathrm{gl.dim}(\mathcal{A}) \geq \mathrm{gl.dim}(\mathcal{B})$.

Proof. Easy consequence of Corollary 1.5. ■

Example 1.7. The 'composition' of split pairs need not be a split pair. Indeed, let R be any ring with a ring endomorphism f which is not surjective. Let $A := R \oplus R$ be the sum of two copies of R and $B := R$. The map $(1, f)$ is an embedding of B into A which, composed with the projection onto the first summand, gives the identity. This induces an exact split pair $F = {}_BA \underset{A}{\otimes} -$ and $G = {}_AB \underset{B}{\otimes} -$ of functors (it is an example of a split quotient as defined in the next section).

Swapping the two summands of A, that is, multiplying by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is an automorphism, which induces an autoequivalence H. The pair (H, Id) also is a split pair.

However, the composition $F \circ H \circ Id \circ G$ sends R to an R-module, which as a set also is R, but with R-action given by f. This module may, for example, decompose. To get an explicit example, choose R to be $k[x]/(x^3)$ for some field k. Let f send x to x^2. Then under the action via f, R decomposes into a two-dimensional summand and a one-dimensional summand.

2. Some exact split pairs

We will describe three classes of examples of exact split pairs. In the next section, we will show that all exact split pairs are made up (in a sense to be made precise) of these three types of examples.

2.1. Split quotients

Let A and B be two rings. Then we call B a *split quotient* of A, if B is a subring of A (via an embedding ε sending the unit of B to that of A), and there exists a surjective homomorphism $\pi\colon A \twoheadrightarrow B$ such that the composition $\pi \circ \varepsilon$ is the identity on B. The homomorphisms π and ε induce two exact functors $F = {}_BA \underset{A}{\otimes} -$ and $G = {}_AB \underset{B}{\otimes} -$ between the categories A-mod and B-mod. The composition $F \circ G$ is the identity on B-mod.

Split quotients are *retracts* of rings. They also appear under the name of *cleft extensions*, for example in [4], where they also have been used to compare cohomology of two module categories.

2.2. Centralizer subrings eAe

Let A be a ring, e an idempotent in A. Let B be the centralizer subring eAe, and let ${}_BT_A$ be the bimodule ${}_BeA_A$ and ${}_AS_B$ the bimodule ${}_AAe_B$. Assume S to be eAe-projective. Then the functors $F = {}_BT_A \underset{A}{\otimes} -\colon A\text{-mod} \to B\text{-mod}$ and $G = {}_AS_B \underset{B}{\otimes} -\colon B\text{-mod} \to A\text{-mod}$ form an exact split pair of functors between A-mod and B-mod.

Sometimes, centralizer subrings eAe are called *corner rings*.

2.3. Morita equivalences

Let A and B be two Morita equivalent rings and let $\varphi\colon A\text{-mod} \to B\text{-mod}$, $\psi\colon B\text{-mod} \to A\text{-mod}$ be two mutually inverse equivalences of categories. Obviously, both (φ, ψ) and (ψ, φ) are split exact pairs of functors, since $\varphi\psi \cong id_{\mathcal{B}}$ and $\psi\varphi \cong id_{\mathcal{A}}$. Both the functor F and the functor G send simple modules to simple modules.

3. All exact split pairs

In this section, we describe the exact split pairs in general. Combining the examples of split quotients and centralizer rings and relaxing the condition in the latter case, we get a more general example of an exact split pair. We then show that up to certain Morita equivalences, this is the general case.

Definition 3.1. Let A be a ring, e an idempotent, and B a split quotient of eAe (viewed as a subring of eAe). Then we call B a *corner split quotient* if there is an A-eAe-bimodule S, which satisfies $eS \simeq B$ as a B-bimodule.

Note that every B-module is an eAe-module via the quotient map. Thus, in the definition, we may equivalently require S to be a right B-module.

Lemma 3.2. *Let B be a corner split quotient of A. Then the functors $F = eA \otimes_A -$ and $G = S \otimes_{eAe} B \otimes_B -$ form an exact split pair.*

Proof. The functor F is exact by construction and G is so by assumption. The composition $F \circ G$ is tensoring with $eA \otimes_A S = eS \simeq B$ as a bimodule; hence it is the identity of B-mod. ■

As Example 1.7 has shown, composing exact split pairs (or even just split quotients) with Morita equivalences in general need not result in a split pair. There are, however, more restricted options of composing split pairs with equivalences in order to produce new split pairs.

Lemma 3.3. *Let A and B be two left Noetherian rings. Let (F, G) be an exact split pair of functors between A-mod and B-mod.*

$$A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \qquad F \circ G \text{ autoequivalence of } B\text{-}mod.$$

(a) *Let $E_1 : A$-mod $\to A'$-mod and $E_2 : A'$-mod $\to A$-mod be two mutually inverse equivalences. Then $(F \circ E_2, E_1 \circ G)$ is an exact split pair.*

(b) *Let $E_3 : B$-mod $\to B'$-mod and $E_4 : B'$-mod $\to B$-mod be any two equivalences. Then $(E_3 \circ F, G \circ E_4)$ is an exact split pair.*

Note that in part (b), we may as well assume $B' = B$ (and hide the equivalence inside B-mod).

If we were to use a more restricted definition of split pairs, requiring $F \circ G$ to be the identity, then the composition of split pairs always would be a split pair.

Proof. We note that equivalences are automatically exact. Then (a) follows from the equality $F \circ E_2 \circ E_1 \circ G = F \circ G$, whereas (b) follows from $E_3 \circ F \circ G \circ E_4$ being an equivalence. ■

Now we can show that the sufficient conditions for exact split pairs given in the previous two lemmas are also necessary; that is, any exact split pair is obtained from a split corner quotient as in Lemma 3.2 by composing with admissible Morita equivalences as in Lemma 3.3.

Theorem 3.4. *Let A and B be two left Noetherian rings. Let (F, G) be an exact split pair of functors between A-mod and B-mod.*

$$A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \qquad (F \circ G \text{ autoequivalence of } B\text{-}mod).$$

Then there exists a ring A', an idempotent $e \in A'$, a bimodule ${}_{A'}S_{eA'e}$ and a pair of mutually inverse equivalences (E_1: A-mod $\to A'$-mod, E_2: A'-mod $\to A$-mod) such that the following properties are satisfied:

The ring B is a split corner quotient of A' with respect to the bimodule S. In particular, B is a split quotient of $eA'e$.

Setting $E_3 = Id : B$-mod $\to B$-mod and $E_4 = (F \circ G)^{-1}$, the following diagram describes the situation:

$$A'\text{-}mod \underset{E_1}{\overset{E_2}{\rightleftarrows}} A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \underset{E_4}{\overset{E_3}{\rightleftarrows}} B\text{-}mod \qquad \begin{array}{ll} F' = E_3 \circ F \circ E_2, & G' = E_1 \circ G \circ E_4 \\ F' \circ G' = id_{B\text{-}mod}. & \end{array}$$

Here, $F' = E_3 \circ F \circ E_2$ and $G' = E_1 \circ G \circ E_4$ are the functors ${}_B eA' \underset{A'}{\otimes} -$ and ${}_{A'}S \underset{eA'e}{\otimes} B \underset{B}{\otimes} -$ respectively, both ${}_B eA'_{A'}$ and ${}_{A'}S \underset{eA'e}{\otimes} B_B$ being right projective and $eS_B \cong B$.

Conversely, any such situation describes an exact split pair.

Proof. Since the functors F and G are exact, there exist two modules ${}_BT'_A$ and ${}_AS'_B$, both right projective, such that $F = {}_BT' \underset{A}{\otimes} -$ and $G = {}_AS' \underset{B}{\otimes} -$. Moreover, $F \circ G$ being an equivalence, we have that ${}_BT' \underset{A}{\otimes} S'$ is a projective generator in B-mod.

The module T'_A is projective, but it may not be of the form eA for any idempotent $e \in A$, since direct summands of T'_A may have multiplicities too large. Therefore, we pass to another algebra A' that is Morita equivalent to A and that has large enough multiplicities of indecomposable projective modules in the regular representation. For instance, we may choose A' to be a matrix algebra over A. In formal terms this means that there exists a ring A', Morita equivalent to A, and mutually inverse equivalences $A'\text{-}mod \underset{E_1}{\overset{E_2}{\rightleftarrows}} A\text{-}mod$ such that the module $T = E_1(T')$ is of the form eA' for some idempotent $e \in A'$.

Recall that we have set $E_3 := Id : B\text{-mod} \to B\text{-mod}$ and $E_4 := (F \circ G)^{-1}$, so we have the diagram as claimed.

$$A'\text{-}mod \underset{E_1}{\overset{E_2}{\rightleftarrows}} A\text{-}mod \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}mod \underset{E_4}{\overset{E_3}{\rightleftarrows}} B\text{-}mod \qquad \begin{array}{l} F' = E_3 \circ F \circ E_2, \qquad G' = E_1 \circ G \circ E_4 \\ F' \circ G' = id_{B\text{-}mod} \end{array}$$

Hence, we have moved into the following situation. We have exact functors $F' = E_3 \circ F \circ E_2$ and $G' = E_1 \circ G \circ E_4$ with $F' \circ G' = id_{B-mod}$. These functors can be written $F' = {}_BT \underset{A'}{\otimes} -$ and $G' = {}_{A'}S \underset{B}{\otimes} -$, with ${}_BT_{A'} = eA'$ and ${}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$. The last isomorphism ${}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$ also identifies right B-module structures. Indeed, tensoring on the left does not affect right module structures and the equivalence $F' \circ G'$ transports the right module structure over the endomorphism ring, which is B, into a right module structure over the image of the endormorphism ring, which is again B.

It remains to check that B is a split quotient of $eA'e$, in a natural way:

Since T is a B-A'-bimodule, there exists a ring homomorphism $\varepsilon\colon B \to \mathrm{End}(T_{A'}) = eA'e$.

Since S is an A'-B-bimodule, there is an $eA'e$-B-bimodule structure on eS. Hence, there is a ring homomorphism $\pi\colon eA'e \to \mathrm{End}(eS_B) = \mathrm{End}(eA' \underset{A'}{\otimes} S_B) = \mathrm{End}(B_B) = B$.

We claim that $\pi\varepsilon = id_B$. Given any $b_1 \in B$, we can write it as eae for some $a \in A'$. We have to show that in

$$\begin{array}{l} B \xrightarrow{\varepsilon} eA'e \xrightarrow{\pi} B \\ b_1 \mapsto eae \mapsto b_2, \end{array}$$

there is an equality $b_1 = b_2$.

Now $eae = \varepsilon(b_1)$ means $eaet = b_1t$ for every $t \in T$, while $b_2 = \pi(eae)$ means that for every $s \in S$, one has $eae \bullet (e \otimes s) = b_2(e \otimes s)$. Here the action of eae on $eA' \otimes S$, denoted by $\bullet$, is given by the action of A' on S under the isomorphism $eS \to eA' \underset{A'}{\otimes} S$ $(es \mapsto e \otimes s)$. Hence, it is given by $eae \bullet (e \otimes s) = e \otimes aes$ and then extended by linearity. The action of B on $eA' \underset{A'}{\otimes} S \cong {}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$ is given by considering B as the endomorphism ring of B_B. Therefore, B is acting on the left with the usual action on the regular module.

Therefore for every $ea' \otimes s \in {}_BT \underset{A'}{\otimes} S_B \cong {}_BB_B$ we have $b_1(ea' \otimes s) = (b_1ea') \otimes s = (eaea') \otimes s = e \otimes (aea's) = eae \bullet (e \otimes a's) = b_2(e \otimes a's) = b_2(ea' \otimes s)$. Thus $b_1 = b_2$ since they act equally on every element of the regular module.

To get the converse of this characterization, we just combine Lemmas 3.2 and 3.3. ■

The following examples show that the theorem cannot be strengthened any further:

Example 3.5. The module S in the definition of split corner quotient need not be projective as a left A-module. In particular, S need not be isomorphic to Ae.

Let k be a field. Let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ and let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $B = eAe = k$ is a split corner quotient when setting $S = Ae$, but also when using the simple A-module $S = Ae/\mathrm{rad}(Ae)$.

While the composition $F \circ G$ is the same in both cases, the images of B-modules under G are different.

Example 3.6. It may even happen that $S = Ae$ does not satisfy our assumptions, but still there is a split corner quotient for a different choice of S.

Let k be a field and $B := k[x]/(x^2)$. Let $A = \begin{pmatrix} k & k \\ 0 & B \end{pmatrix}$(where multiplication uses the action of B on its simple quotient k). Let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $B = eAe = k$ is a split corner quotient when setting $S = Ae/\mathrm{rad}(Ae)$, but Ae is not projective as a right B-module.

Example 3.7. Often, a good candidate for S is $S = Ae \otimes_{eAe} B$ for some split quotient B of eAe. In this situation, the condition ${}_BeS = {}_BB$ is automatic, and it remains to check that S is right B-projective.

In the context of Brauer algebras (Section 4.2), we will see examples where S takes this form $Ae \otimes_{eAe} B$ for some B strictly smaller than eAe. Note that $Ae \otimes_{eAe} B$ is (in general, and usually in these examples) not isomorphic to the restriction of the right eAe-module structure of Ae to B.

4. Examples of exact split pairs

In this section, we collect a number of situations in the literature to which our machinery applies in a natural way. In some cases, we thus re-prove known results, and in other cases we get something new.

We first list some well-known examples of split quotients.

A semidirect product of finite groups fits into a split quotient situation relating the group algebra of the quotient subgroup with that of the semidirect product.

Let R be a ring and $R[x]$ be the polynomial ring over R in the indeterminate x. (That is, x is like a loop.)

The ring homomorphisms $\varepsilon\colon R \to R[x]$ (the canonical embedding) and $\pi\colon R[x] \to R$ (the canonical projection on the zero-degree term) show that R is a split quotient of $R[x]$.

Thus, there exists a split exact pair of functors between $R[x]$-mod and R-mod. Similar split pairs exist for various 'twisted' polynomial rings.

In a similar way, one can show there exists a split exact pair of functors between $R\,[[x]]$-mod and R-mod, where $R\,[[x]]$ is the ring of formal power series over R.

Note that the generator does not need to be torsion-free. The ring R is a split quotient of $R[x]/\,(f(x))$ as well, if $f(x)$ has no zero-degree term.

4.1. Tensor products and twisted tensor products

Our technology applies both to tensor products of algebras and to algebras which are tensor products of other algebras, but with slightly twisted multiplicative structures.

First we deal with the classical case:

Proposition 4.1. *Let A and B be finite dimensional algebras over a perfect field k. Then there exists a split pair relating $A \otimes_k B$ and A.*

Proof. Let $S(B)$ be a maximal subalgebra of B. Then there is a split quotient situation $S(B) \hookrightarrow B \twoheadrightarrow S(B)$, which induces a split quotient situation $A \otimes_k S(B)$. Another split quotient situation relates the semisimple algebra $S(B)$ with k by combining a Morita equivalence to a product of copies of k with projection onto one component. ■

A similar argument also works in a much more general situation, such as, for example, covering algebras, which have attracted some recent interest within the theory of quasi-hereditary algebras.

Proposition 4.2. *Let A be a finite-dimensional algebra over a perfect field k, which has a vector space decomposition $A = B \otimes_S C$ such that B and C are k-algebras, S is a semisimple algebra contained in a maximal semisimple subalgebra $S(B)$ of B, and also in a maximal semisimple subalgebra $S(C)$ of C, and B and C are subalgebras of A via $B \simeq B \otimes_S S \subset B \otimes_S S(C) \subset B \otimes_S C = A$ and $C \simeq S \otimes_S C \subset S(B) \otimes_S C \subset B \otimes_S C = A$.*

Then there are split pairs (A, B) and (A, C).

Proof. As in the previous proof, there are split quotient situations $B \otimes_S S(C) \subset A$ and $S(B) \otimes_S C \subset A$, which can be combined with split quotients relating $S(B)$, or $S(C)$, and S and then Morita equivalences $B\text{-}mod \simeq B \otimes_S S\text{-}mod$ and $C\text{-}mod \simeq S \otimes_S C\text{-}mod$. ■

Algebras of this kind include Xi's dual extension algebra [19] and the twisted doubles of Deng and Xi [7]; these constructions (imposing additional conditions on B and C) were defined to produce examples of quasi-hereditary algebras which could then be related to subalgebras B and C. In most of the situations studied, much stronger statements are true than what our machinery is producing, but in more general situations (still covered by these definitions), we get new results.

4.2. Brauer algebras

In this subsection, we find split corner quotients 'in nature', relating the cohomology of Brauer algebras (which occur in the representation theory of algebraic groups of type B and C) with cohomology over symmetric groups (which occur in type A). On the ring theoretic level, Brauer algebras have been related to symmetric groups in [13] in terms of cellular structures. On a module theoretic level, such connections have been found and used in [9], where also some cohomological statements can be found. Our results add another cohomological aspect to this connection between types A and B, C.

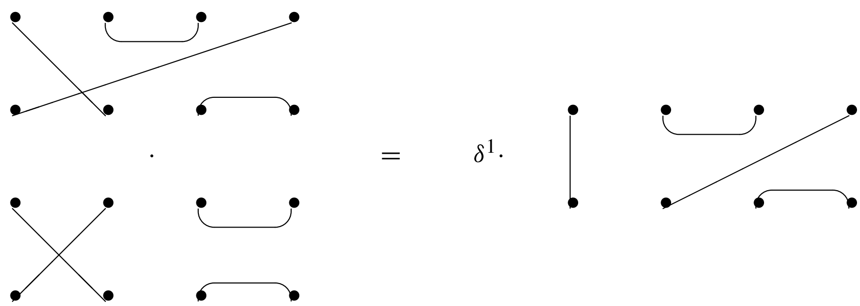
The Schur–Weyl duality relates the representation theory of the infinite group $GL_n(k)$ (where k is an infinite field of arbitrary characteristic) with that of the symmetric group Σ_r via the mutually centralizing actions of the two groups on the space $(k^n)^{\otimes r}$. Brauer defined the algebras that are now called 'Brauer algebras by an analogous situation, where GL_n is replaced by either an orthogonal or a symplectic group (types B and C) and the group algebra of the symmetric group is replaced by a Brauer algebra. More precisely, for a fixed integer r and a given base ring k (a field in Brauer's case), a whole family of Brauer algebras $B_k(r, \delta)$ is defined, depending on a parameter $\delta \in k$, which has to be specialized to certain integers to cover the situation Brauer was interested in.

More recently, Brauer algebras and their generalizations, especially the Birman–Murakami–Wenzl algebras, have been looked at in the context of quantum groups and low-dimensional topology. Other closely related algebras, such as Temperley–Lieb algebras and partition algebras, are also of interest in statistical mechanics.

Definition 4.3. Fix a commutative Noetherian domain k, an element $\delta \in k$ and a natural number r. Then the *Brauer algebra* $B_k(r, \delta)$ as a free k-module has a basis consisting of diagrams of the following form: a diagram contains $2r$ vertices, r of them called 'top vertices' and the other r called 'bottom vertices' such that the set of vertices is written as a disjoint union of r subsets, each of them having two elements; these subsets are called 'edges'. Two diagrams x and y are multiplied by concatenation, that is, the bottom vertices of x are identified with the top vertices of y, thus giving rise to edges from the top vertices of x to the bottom vertices of y, hence defining a diagram z. Then $x \cdot y$ is defined to be $\delta^{m(x,y)} z$, where $m(x, y)$ counts those connected components of the concatenation of x and y which do not appear in z–that is, which neither contain a top vertex of x nor a bottom vertex of y.

(Note that in this definition and for the rest of this section, k need not be a field, but any commutative Noetherian domain.)

Let us illustrate this definition by an example, multiplying two elements in $B_k(4, \delta)$:



Brauer algebras are cellular algebras [8,13]; in particular, they have cell modules, which play a role analogous to that of Specht modules for symmetric groups.

An easy observation is:

Proposition 4.4. *The group algebra $k\Sigma_r$ is a split quotient of $B(r, \delta)$.*

Proof. Those diagrams which just consist of through strings (that is, strings going from the top row to the bottom row) define permutations, and the free k-module generated by them is a subalgebra of $B(r, \delta)$, which is isomorphic to $k\Sigma_r$. Those diagrams, which have at least one horizontal edge (in the top row and thus also in the bottom row) are the k-basis of a two-sided ideal, the quotient by which again is $k\Sigma_r$, and this quotient map restricts to an isomorphism on the algebra generated by through strings. ∎

At this point we get for free the following known corollary:

Corollary 4.5 ([14]). *Let k be a field, $\delta \neq 0$ and $\mathrm{char}(k) = p$. Then the Brauer algebra $B(r, \delta)$ has finite global dimension if and only if $p > r$.*

Proof. (Note that $B(r, \delta)$ is rarely symmetric or self-injective, unlike $k\Sigma_r$. Thus the problem is non-trivial.) If $p > n$, then the cell chain given in [13] is a heredity chain. Thus $B(r, \delta)$ is a quasi-hereditary algebra, and hence of finite global dimension. (See [13,14] for details.) If $\mathrm{char}(k) = p \leq n$, then the known cell chain is not a heredity chain, and at this point one may invoke the main theorem of [14] to conclude that $B(r, \delta)$ must have infinite global dimension. Alternatively, and more easily, this follows from Proposition 1.6 by using the fact that for $\mathrm{char}(k) = p \leq r$, the group algebra $k\Sigma_r$ has infinite global dimension. ∎

A more interesting application, which makes full use of our split pair technology, providing non-trivial examples of split corner quotients, is a formalization of the observation that the Brauer algebra $B(r, \delta)$ is related not just to the symmetric group algebra $k\Sigma_r$, but also to many smaller symmetric groups. Indeed, in [13], Theorem 5.6, the Brauer algebra has been written as follows:

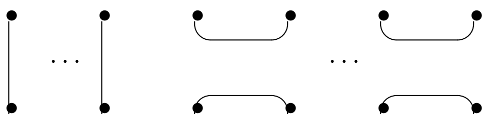
As a free k-module, $A = B(r, \delta)$ is equal to

$$k\Sigma_r \oplus (V_{r-2} \otimes V_{r-2} \otimes k\Sigma_{r-2}) \oplus (V_{r-4} \otimes V_{r-4} \otimes k\Sigma_{r-4}) \oplus \dots$$

(ending with indices 0 or 1 when r is even or odd), where V_l is a free k-module, whose k–rank equals the number of possibilities to draw $(r - l)/2$ edges between $r - l$ out of r vertices. This decomposition produces a chain of ideals (which can be refined to a cell chain) of $B(r, \delta)$, where the ideals are defined by adding up any right hand part $(V_l \otimes V_l \otimes k\Sigma_l) \oplus (V_{l-2} \otimes V_{l-2} \otimes k\Sigma_{l-2}) \oplus \dots$ in this decomposition.

Each layer $V_l \otimes V_l \otimes k\Sigma_l$ (which is a subquotient of two ideals in the above chain of ideals) has a basis consisting of diagrams with $(r - l)/2$ horizontal edges in top and bottom row each (recorded in the first and second copy of V_l), and the remaining edges being through strings (recorded as elements of the symmetric group Σ_l).

From now on, let us assume that δ is invertible in k. Then we define (as in [13] or [9]) an idempotent element e_l in $A = B(r, \delta)$ by $e_l = \delta^{-(r-l)/2} d_l$, where d_l is the diagram obtained by putting l vertical through strings at the beginning, and then putting horizontal edges relating to a vertex with a direct neighbor, that is, d_l is of the form:



The corner ring $e_l A e_l$ is isomorphic to the Brauer algebra $B(l, \delta)$. It has, of course, a split quotient $A_l \simeq k\Sigma_l$.

Proposition 4.6. *Using the above notation (in particular, δ is invertible), the algebra $A_l \simeq k\Sigma_l \subset e_l A e_l$ is a corner split quotient of the Brauer algebra $B(r, \delta)$.*

Proof. This proof is very similar to arguments used in [9], where a general theory of Young modules for Brauer algebras is developed.

We know already that $A_l \simeq k\Sigma_l$ is a split quotient of the small Brauer algebra $B(l, \delta)$, which is isomorphic to the corner ring $e_l A e_l$ of the big Brauer algebra $A = B(r, \delta)$ and we also know the ring homomorphisms used in this context. It remains to prove that the module $Ae_l \otimes_{e_l A e_l} A_l$ is projective as a right A_l-module.

By the multiplication rule in the Brauer algebra, and by the definition of e_l, the projective A-module Ae_l has a basis consisting of diagrams with at least $m = (r - l)/2$ horizontal edges, where the bottom row has at least the horizontal edges occuring in e_l. Similarly, $e_l A e_l$ has a basis consisting of diagrams with at least m horizontal edges, where both in the top and in the bottom row, at least the horizontal edges used in e_l do occur. The algebra $e_l A e_l$ acts on A_l via the quotient map α, which has in its kernel all diagrams in $e_l A e_l$ with more than m loops; that is, if a diagram has loops not already in e_l, then the diagram is in the kernel of α.

The tensor product $Ae_l \otimes_{e_l A e_l} A_l$ is generated (over k) by tensors of the form $x \otimes y$, where x is a diagram sharing the m horizontal edges in its bottom row with e_l, but possibly having more of them, and y is an element in the symmetric group Σ_l. If x has more than m horizontal edges, then we can write $x = x \cdot e_q$ for some $q < l$, with $e_q \in e_l A e_l$. Here, e_q is an idempotent in a lower layer of the cell chain, having more than m horizontal edges in each row. Therefore, $\alpha(e_q) = 0$. Thus $x \otimes y = x \cdot e_q \otimes y = x \otimes e_q y = 0$. Therefore, $Ae_l \otimes_{e_l A e_l} A_l$ is generated (over k) by elements $x \otimes y$ with x having precisely m horizontal edges in each row, those in the bottom row being the same as in e_l, whereas those in the top row can be arranged freely. Rewriting, by a slight abuse of notation, $x \otimes y$ as $xy \otimes 1$ (with 1 the unit in the symmetric group Σ_l), it follows that $Ae_l \otimes_{e_l A e_l} A_l$ is just a direct sum of copies of A_l (the number given by the number of possibilities to arrange the m horizontal edges in the top row of x, that is, by the dimension of V_l). Indeed, let $J \subset Ae_l$ be the left ideal generated (over k) by diagrams with more than m edges. (This is the intersection with Ae of an ideal in the cell chain.) Then we have just shown that $J \otimes_{e_l A e_l} A_l$ vanishes. Hence, $Ae_l \otimes_{e_l A e_l} A_l$ is isomorphic to $(Ae_l/J) \otimes_{e_l A e_l} A_l$. Those elements in $e_l A e_l$, which have more than m horizontal edges in each row, act trivially both on A_l and on Ae/J. Thus the tensor product $A/J \otimes_{e_l A e_l} A_l$ over $e_l A e_l$ is isomorphic to the tensor product $A/J \otimes_{A_l} A_l$ over A_l, which leaves us with A/J. This has a k-basis consisting of diagrams which, in the bottom row, have m horizontal edges arranged in the same way as in e_l and the m horizontal edges in the top arranged freely. The algebra A_l acts on the right by the symmetric group's action on the through strings. ■

We note that in this situation, Ae_l need not be projective as right $e_l A e_l$-module. (The case $r = 4$, $l = 2$ produces already a counterexample.)

We refer the reader to [9] for more details on comparing A-modules with A_l-modules, especially cell modules and Young modules.

Finally, we note that similar situations occur for other diagram algebras, such as partition algebras or Birman–Murakami–Wenzl algebras.

5. Homological reductions and the strong no loops conjecture

In this section, we work with finite dimensional algebras $A = kQ/I$ given by a quiver Q and a relation ideal $I = \langle R \rangle$; k is any field. We are collecting some reduction methods relating cohomology in A-mod to that in module categories of smaller algebras, defined by removing parts of the quiver Q. The aim is to get lower bounds for cohomology of A-modules. At the end of the section, we apply these lower bounds to obtain the validity of the strong no loops conjecture for certain classes of algebras.

The setup is the following: let k be a field, $Q = (\Delta_0, \Delta_1)$, a quiver and kQ the path algebra. Let R be a set of relations (linear combinations of paths) in kQ, and $I = \langle R \rangle$ the relation ideal in kQ generated (as a two-sided ideal) by R. Let $A = kQ/I$ be the path algebra of Q over k with relations R.

5.1. Removing vertices, keeping cohomology

This subsection does not use our machinery of split pairs. We just quote and then apply results from the literature, about isomorphisms in cohomology.

The context is that of an algebra A and a quotient algebra B Modulo an ideal J satisfying strong properties. The following theorem is due to Cline, Parshall and Scott (cf. [6], Theorem 3.1). It has been crucial in the development of the theory of quasi-hereditary algebras and, more generally, of stratified algebras.

Theorem 5.1 (*[6], Theorem 3.1*). *Let A be a ring, J be an ideal of A and $B = A/J$. Let $\mathcal{A}$ be the category of all left A-modules and $\mathcal{B}$ the category of all left B modules. The full embedding $i: \mathcal{B} \to \mathcal{A}$ induces a functor between the derived categories $i_*: D^b(\mathcal{B}) \to D^b(\mathcal{A})$. This functor is a full embedding if:*

(a) $\mathrm{Ext}^n_A({}_A B, {}_A B) = 0$ *for every $n > 0$ and*
(b) $\mathrm{pdim}({}_A B) < \infty$.

We are applying this result in a very concrete situation; our aim is to remove certain vertices v from the quiver Q of $A = kQ/I$.

Let v be a vertex in Q. We will consider the quiver $Q^v = (\Delta_0^v, \Delta_1^v)$ obtained from Q by removing v and the arrows starting from v or ending in v. In the path algebra kQ^v, we define the ideal $I^v = \{r \in I \mid$ no summand of r passes through $v\}$. Let us denote by A^v the algebra kQ^v/I^v.

If the vertex v is a source or a sink, the algebra A^v is isomorphic to the algebra A/J, where J is the ideal Ae_vA generated by the idempotent e_v associated to the vertex v. (Note that in our notation, a projective module Ae is k-generated by all paths ending at the vertex e. Thus a source e has a simple projective module Ae.)

In order to apply the theorem, it is sufficient to show that the ideal J is projective and that $\mathrm{Ext}^1_A({}_AB, {}_AB) = 0$.

If v is a sink, then $J = Ae_vA = Ae_v$ (since no non-trivial path leaves v) is projective, and there are no nonzero homomorphisms ${}_AJ \to {}_AB$. This implies that $\mathrm{Ext}^1_A(B, B) = 0$ since it is a quotient of $\mathrm{Hom}_A(J, B) = 0$.

If v is a source, then $J = Ae_vA$ is the trace of the simple projective A-module Ae_v; hence it it is a semisimple projective module. Finally $\mathrm{Ext}^1_A({}_AB, {}_AB) = 0$, since it is a quotient of $\mathrm{Hom}({}_AJ, {}_AB)$ which vanishes by definition of J and of B.

Therefore, the functor i_* is a full embedding, thus giving isomorphisms, not just epimorphisms, between Ext groups in A-mod and B-mod.

This gives the first two statements in the following proposition (which also could be proved directly, not using [6], without much effort):

Proposition 5.2. *Keep the above notation for $A = kQ/I$ and A^v.*

(a) *Suppose v is a sink and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.*

(b) *Suppose v is a source and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.*

(c) *Suppose v is a sink but for loops (that is, all arrows ending in v are loops at v) and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has them over A.*

(d) *Suppose v is a source except for loops (that is, all arrows leaving v are loops at v), and L is a simple B-module. Then L has non-vanishing self-extensions $Ext^n(L, L)$ over B in infinitely many degrees n if and only if it has so over A.*

Proof. The first two statements have already been shown. In case (c), the two-sided ideal $J = Ae_vA = Ae_v$ is again projective, as a left A-module, and the same proof works as for (a).

Denote by A^{op} the opposite algebra of A and by L' the simple A^{op}-module corresponding to L under the duality $\mathrm{Hom}_k(-, k)$. Using the isomorphism $\mathrm{Ext}^n_A(L, L) \simeq \mathrm{Ext}^n_{A^{\mathrm{op}}}(L', L')$, and noting that v is a sink in the quiver of A^{op}, claim (d) follows from (c). ■

We remark that in the situation of the proposition, stronger statements are true. In parts (a)–(c) it is true that $\mathrm{Ext}^n_A(X, Y) \simeq \mathrm{Ext}^n_B(X, Y)$ for all n and all B-modules X and Y. Since J in these three cases is projective, we can also say that the projective dimension of a B-module X is finite if and only if X has finite projective dimension over A.

The proposition does not however, give any information about the cohomology between a simple B-module S_1 and a simple A-module S_2, which is not defined over B. Therefore, in part (d), it also does not relate the projective (or in part (c) the injective) dimension of S_1 over B to the same dimension over A. The following example shows that these dimensions can be rather different.

Example 5.3. Let k be a field and $B := k[x]/(x^2)$. Let $A = \begin{pmatrix} B & k \\ 0 & k \end{pmatrix}$(where multiplication uses the action of C on its simple quotient k). Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and let v be the vertex associated with e; v is a source but for loops. Here, $A^v = k$. But over A, the simple A^v-module S has infinite projective dimension.

5.2. Removing arrows, reducing cohomology

As before, we are given an algebra $A = kQ/I$ by a quiver and relations. We use split quotients to remove arrows from the quiver Q.

Let α be an arrow in Q. We consider the quiver $Q^\alpha = (\Delta_0, \Delta_1 \setminus \{\alpha\})$ obtained from Q by removing α. In the path algebra kQ^α, we consider the ideal $I^\alpha = \{r \in I \mid$ no summand of r has α as a subpath $\}$. Let us call A^α the algebra kQ^α/I^α. For every A-module M associated to the representation $(V_i, \varphi_\beta)_{i\in\Delta_0,\beta\in\Delta_1}$ of the quiver Q which respects the relations in $I = \langle R\rangle$, we can consider the representation of the quiver Q^α given by $(V_i, \varphi_\beta)_{i\in\Delta_0^\alpha,\beta\in\Delta_1^\alpha}$. Obviously, this representation respects the relations in I^α, and therefore it is associated to an A^α module M^α. The map $M \to M^\alpha$ (extended to morphisms in the obvious way) defines a functor between the Abelian categories A-mod and A^α-mod. We will denote it by F^α. This is an exact functor.

Proposition 5.4. *Keep the above notations.*

(a) *The algebra A^α is isomorphic to the subalgebra of A, which is generated by the set $\{p + I \in A = kQ/I \mid p$ a path in Q and α not a subpath of $p\}$.*

(b) *Assume that the arrow α is involved only in monomial relations. Then $A/A\alpha A \cong A^\alpha$ and the surjective homomorphism $A \to A^\alpha$ induces a full embedding G: A^α-mod $\to$ A-mod. Then (F^α, G) is an exact split pair of functors.*

Note that we may remove more than one such arrow at a time, since the composition of split quotients is again a split quotient.

Proof. In the path algebra kQ, the two-sided ideal $kQ \cdot \alpha \cdot kQ$ is generated (over k) by all paths going through α. The subalgebra kQ^α in whose quiver α is missing, is a split quotient of kQ via the projection $kQ \to kQ/(kQ \cdot \alpha \cdot kQ)$.

We can write $I^\alpha = I \cap kQ^\alpha$. Indeed, I^α is contained in the right hand side by definition. Conversely, an element $r \in I - I^\alpha$ has a summand, which has the arrow α as a subpath. Hence $r \notin kQ^\alpha$.

The subalgebra A^α of A is a quotient of kQ^α. In fact, A^α is the image of kQ^α under the projection from kQ to A. The relation ideal of A^α is $I \cap kQ^\alpha = I^\alpha$.

If α is involved only in monomial relations, then any of the generating relations r is either a path containing α or a linear combination of paths, none of which contains α. Hence we can decompose the relation ideal I into a direct sum $I = I^\alpha \oplus ((kQ)\alpha(kQ) \cap I)$, where I^α as above is k-generated by all relations not involving α. Then $A^\alpha \simeq kQ^\alpha/I^\alpha \simeq (kQ^\alpha \oplus (kQ\cdot\alpha\cdot kQ))/(I^\alpha \oplus (kQ\cdot\alpha\cdot kQ)) \simeq (kQ^\alpha \oplus (kQ\cdot\alpha\cdot kQ))/(I + (kQ\cdot\alpha\cdot kQ)) \simeq A/A^\alpha$. ■

Note that if α is involved in non-monomial relations, then the algebra A^α exists, but it may not be isomorphic to a quotient of A any more. The functor F^α as explicitly constructed is still an exact functor between the categories A-mod and A^α-mod. It may, however, lack a right inverse and it may not be surjective on the morphisms.

5.3. Removing vertices and reducing cohomology

In order to remove vertices which are neither sinks nor sources (and all arrows and loops attached to these vertices), we will use corner rings.

Let $\Delta_0 = \{1, 2, \ldots, i, \ldots, n\}$ be the set of vertices of Q, let e_j, for every j in Δ_0, be the trivial (idempotent) path starting and ending at the vertex j, and let $e = e_1 + e_2 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_n = 1 - e_i$.

Let Q' be the quiver with vertices $\Delta'_0 = \Delta_0 \setminus \{i\}$ and arrows $\Delta'_1 = \{\alpha \in \Delta_1 \mid e(\alpha) \neq i \neq s(\alpha)\} \cup \{p = \alpha_n \ldots \alpha_1$ path in$Q \mid s(\alpha_1) \neq i \neq e(\alpha_n) \wedge s(\alpha_n) = \cdots = s(\alpha_2) = i = e(\alpha_{n-1}) = \cdots = e(\alpha_1)\}$. Thus $kQ' = ekQe$.

Given R a set of relations over kQ such that $A = kQ/\langle R\rangle$, we define a set of relations R' such that $\langle R\rangle = \langle R'\rangle \cap kQ'$: Write $R = R_0 \cup R_e \cup R_s \cup R_{\text{se}}$ where R_0 is the set of the relations in R neither starting nor ending at the vertex i, R_s is the set of the relations starting, but not ending, at i, R_e is the set of the relations ending, but not starting, at i, and R_{se} is the set of the relations both starting and ending at the vertex i. Consider the set of relations $R' = R'_0 \cup R'_e \cup R'_s \cup R'_{\text{se}}$ on the path algebra kQ', where $R'_0 = R_0$, $R'_s = \{\alpha r \mid r \in R_s, \alpha$ arrow $e(\alpha) = i, s(\alpha) \neq i\}$, $R'_e = \{r\alpha \mid r \in R_e, \alpha$ arrow , $s(\alpha) = i, e(\alpha) \neq i\}$, $R'_{\text{se}} = \{\alpha r\beta \mid \alpha, \beta$ arrows , $r \in R_{\text{se}}, e(\alpha) = i = s(\beta)\}$. We are stretching the notation, since the path $p = \alpha_n \ldots \alpha_1$ has length $\ell > 1$ when considered as an element of R, while it can be an arrow when considered as an element of R'; thus some of the new relations may not be admissible (that is, they may involve paths of length one).

Proposition 5.5. *Keep the above notation.*

(a) *There is an algebra isomorphism $A' = kQ'/\langle R'\rangle \cong eAe$.*

(b) *Suppose $R_s e = \emptyset$, that is, no relation ending in a vertex $j \neq i$ is starting at i. Then A and eAe are related by an exact split pair (with bimodule Ae).*

(c) *Suppose $eR_e = \emptyset$, that is, no relation starting at a vertex $j \neq i$ is ending at i. Then A and eAe are related by an exact split pair (with bimodule eA) for their categories of right modules.*

Proof. Statement (a) is clear by construction.

For claim (b), we need to show that Ae is projective over B. Decompose the right B-module Ae into a projective summand eAe and another summand e_iAe. Then $e_iAe = \oplus_{j\neq i} e_iAe_j$ is k-generated by all paths starting at i and ending in vertices different from i. Fix a vertex $j \neq i$ and let $p_1, \dots p_l$ be the shortest possible paths from i to j; that is, they form a generating set of non-zero paths from i to j, which consist only of loops at i composed with arrows from i to j. Then multiplying with $(p_1, \dots, p_l)$ is an injective B-module map from $(e_jAe)^l$ into e_iAe, since there is no relation involving any of the $p_1, \dots, p_l$. Varying j produces disjoint images under these maps and adding them all up shows e_iAe is projective as a B-module.

(c) follows from (b) by considering opposite algebras. ■

Setting $B = A'$, we can describe the functor $F = {}_BeA_A \underset{A}{\otimes} -\colon A\text{-mod} \to B\text{-mod}$ explicitly in terms of representations. Given a representation $V = (V_j, \varphi_\alpha)_{j\in\Delta_0,\alpha\in\Delta_1}$ of the quiver Q over the field k respecting the relations in R (that is, given an A-module), we get $F(V) = (V_j, {\varphi'}_\alpha)_{j\in{\Delta'}_0,\alpha\in{\Delta'}_1}$, where ${\varphi'}_\alpha = \varphi_\alpha$ if $\alpha \in \Delta_1$, $e(\alpha) \neq i \neq s(\alpha)$ and $\varphi_\alpha = \varphi_{\alpha_n} \circ \cdots \circ \varphi_{\alpha_1}$ if $\alpha = \alpha_n \dots \alpha_1$ with $s(\alpha_1) \neq i \neq e(\alpha_n) \wedge s(\alpha_n) = \cdots = s(\alpha_2) = i = e(\alpha_{n-1}) = \cdots = e(\alpha_1)$.

5.4. Removing parts of the quiver, reducing cohomology

Combining the previous methods allows us to cut larger parts of the quiver:

Proposition 5.6. *Suppose we are given $A = kQ/I$ with $I = \langle R\rangle$ and e an idempotent. Let $A' = eAe$. Denote the vertices involved in e by Δ_1 and the others by Δ_2.*

(a) *Suppose no relation ending at vertices in Δ_1 is starting in a vertex in Δ_2. Then A and $A' = eAe$ are related by an exact split pair.*

(b) *Suppose the arrows from Δ_2 to Δ_1 are involved in monomial relations only. Then A and $A' = eAe$ are related by a sequence of exact split pairs (from A to a split quotient $\tilde{A}$ and then from $\tilde{A}$ to A').*

Both claims have right module analogues as well.

Proof. (a) has the same proof as part (b) of Proposition 5.5.

(b) First, we apply Proposition 5.4 to remove all arrows from Δ_2 to Δ_1. This can be done by just one split quotient (which is the composition of the split quotients used to remove one such arrow at a time). Afterwards, we can apply (a). ■

The relations of A' can be described in an analogous way, as in the case of e being primitive.

5.5. Some cases of the strong no loops conjecture

The reduction methods developed so far can be used to prove the strong no loops conjecture for certain classes of algebras.

The strong no loops conjecture (SNLC) states the following: Let A be an Artinian algebra and S a simple A-module. Suppose, $Ext^1_A(S, S) \neq 0$. Then S has infinite projective dimension. (This is open problem (7) in the list of open problems in [3].)

An algebra A whose underlying quiver has a loop at a vertex v, has an infinite global dimension. This statement, the 'no loops conjecture' (proved by Igusa [10] and implicitly by Lenzing [16]) means that at least one of the simple A-modules has an infinite projective dimension; the strong no loops conjecture says the simple module associated to the vertex v has infinite projective dimension.

Igusa proved in [10] that the strong no loops conjecture holds for monomial algebras, i.e. for finite dimensional algebras which are quotients of a path algebra kQ of a quiver Q over a field k modulo a relation ideal which is generated by a set of paths.

We first list two classes of algebras for which SNLC is true by the methods from Section 5.1; these algebras may serve as input for part (a) of Theorem 5.8.

We call an algebra A *quasi-directed*, if its primitive idempotents can be ordered in such a way that $\mathrm{Hom}_A(Ae_i, Ae_j) \neq 0$ implies $i \leq j$. (Note that there is no condition on endomorphisms of indecomposable projective modules.) Such an algebra has no oriented cycles except possibly loops.

We use the definition of *standardly stratified* algebras given in [1]. (Note that the term 'stratified algebra' is not completely unified in the literature.)

Proposition 5.7. (a) *SNLC holds true for local algebras.*

(b) *SNLC holds true for quasi-directed algebras.*

(c) *SNLC holds true for standardly stratified algebras.*

Proof. A local algebra is either simple or of infinite global dimension, hence (a).

In the cases (b) and (c), we can inductively apply the methods of Section 5.1. In case (b) we use Proposition 5.2.

Part (c) follows from a fundamental property of standardly stratified algebras, which itself is a consequence of Theorem 5.1: The derived category of a standardly stratified algebra A has a stratification (a sequence of recollements) by derived categories of local algebras. Hence we may assume that the simple module S is in the lowest layer of the stratification. In other words, S is the head of a projective standard module $\Delta = Ae$, whose endomorphism algebra eAe is local. There being a loop at S in the quiver of A means eAe is not simple. The composition length $n > 1$ of the local algebra eAe coincides with the composition multiplicity $[\Delta : S]$. Since projective A-modules are filtered by standard modules, and Δ is the only standard module with composition factor S, all composition multiplicities $[P : S]$, if non-zero, are integral multiples of n, for any projective module P. Since $[S : S] = 1$ is not divisible by n, the module S cannot have a finite projective resolution over A. ■

The main result in this section generalizes Igusa's result for monomial algebras. It implies the validity of SNLC for relatively large classes of algebras, and at the same time it constructs more algebras with SNLC from known ones.

We call an algebra A a *monomial union of corner rings* e_iAe_i if the following conditions are satisfied:

1. the idempotents e_i are pairwise disjoint and orthogonal and they add up to the unit of A;
2. the arrows from vertices in e_i to vertices in e_j, for any $i < j$, are involved only in monomial relations.

We call an algebra *quasi-monomial* if its arrows except possibly the loops are involved in non-monomial relations only.

Theorem 5.8. (a) *Let A be a monomial union of corner rings e_iAe_i and suppose SNLC holds true for each e_iAe_i. Then SNLC holds true for A itself.*

(b) *SNLC holds true for monomial algebras.*

(c) *SNLC holds true for quasi-monomial algebras.*

Proof. Part (a) is proved by inductively applying Proposition 5.6. Parts (b) and (c) are special cases of (a).

While the previous results describe globally defined classes of algebras, the next result is local, allowing one to single out certain loops.

Proposition 5.9. *In $A = kQ/I$, choose an idempotent e. Assume that e is not involved in oriented cycles outside of e; that is, assume that the multiplication map $eA(1-e) \otimes_k (1-e)Ae \to eAe$ is zero. Then there is an exact split pair relating A and eAe.*

In particular, if e is primitive and its simple module L satisfies $Ext^1_A(L, L) \neq 0$, then L has infinite projective dimension, that is, it satisfies SNLC.

Proof. We set up a split corner quotient situation with $B = eAe$. Setting $S = Ae$ does not work in general. However, the assumption implies that $(1-e)Ae$ is an A-submodule of Ae; in fact, $eA(1-e)Ae = 0$ implies $A(1-e)Ae \subset (1-e)Ae$. Thus $S = Ae/(1-e)Ae$ equals eS and it is a left A-module, which as a right and left eAe-module is just $B = eAe$ itself. ■

This can be generalized further: For simplicity we give only a statement for a primitive idempotent e.

Theorem 5.10. *In $A = kQ/I$, choose a primitive idempotent e corresponding to the simple module S. Denote the loops at e by $\alpha_1, \ldots, \alpha_l$ ($l \geq 1$)and the oriented cycles at e, which are not loops, by $\beta_1, \ldots, \beta_m$ ($m \geq 0$). Suppose for some $p \geq 1$ the first p loops, $\alpha_1, \ldots, \alpha_p$ are not involved in relations of the form $a = b$, where a is a linear combination of products of these p loops and b involves also loops $\alpha_{p+1}, \ldots, \alpha_l$ or cycles $\beta_1, \ldots, \beta_m$. Then SNLC is true for the simple module S.*

Proof. Let B be the subalgebra of A generated by the loops $\alpha_1, \ldots, \alpha_p$. The assumption guarantees that B is a split quotient of eAe. Let X be the A-submodule of Ae, which is generated by the trace of all Af with f not equivalent to e. Let Y be the A-submodule of Ae, which is generated by the loops $\alpha_{p+1}, \ldots, \alpha_l$. Let $S = Ae/(X + Y)$. By definition, Ae is generated by all paths ending at e. The quotient Ae/X is generated by all paths ending at e and not going through any vertex different from e, that is, by all loops at e. Thus S is generated by all paths, which are just products of the loops $\alpha_1, \ldots, \alpha_p$. Hence $S = eS$, and as a left and right B-module it is isomorphic to B. ■

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